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## To the student

There are college students who have done quite well in mathematics up to and including calculus, but who find that their first encounter with abstract math is a somewhat traumatic experience. A likely reason for these difficulties may be the fact that excelling in the purely computational aspects of math like the plug-and-chug method is not sufficient for writing arguments down and proving theorems in classes like Real Analysis, Abstract Algebra or Topology. Writing proofs is a skill you acquire with lots of practice, similar to being able to quickly substitute, computing a derivative with the chain rule, summing, or factoring a polynomial to find the roots.

In order to be able to read this book we assume that you have knowledge about basic algebra, familiarity of integers, trigonometry, some plane geometry and calculus. Most of you may have seen these subjects in high school or in the first two years of college. In particular, we will be interested in the set of natural numbers  $\mathbb{N} = \{1, 2, 3, \dots\}$ , the set of integers  $\mathbb{Z} = \dots, -2, -1, 0, 1, 2, \dots$ , the set of rationals (quotients of integers  $p/q$  including  $p=0$ ) and  $\mathbb{Q}$  will be the set of real (continuous plus numbers like  $\sqrt{2}$ ) and decimal numbers with non-repeating decimals), which appear in many calculus textbooks. We will focus the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  (functions defined on  $\mathbb{R}$  which takes one value). If the domain of the function is just as intended  $\mathbb{R}$ , we will write  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

Throughout this book, we will emphasize the necessity of proving facts, and we will introduce new symbols and concepts necessary for a transition to more advanced mathematics. You will have the feeling and we will do things all over again, but from a different point of view. For example, you may know that if the square of an integer  $n$  is even, then  $n$  has to be even. But how do you prove that rigorously? You will learn the language of axioms and theorems and you will write convincing and elegant proofs using quantifiers. You will solve many exercises and encounter some challenging projects.

It is common for a college student to ~~not~~ care at all of the professor and perhaps to regard the textbook writer as a ~~foe~~ or some kind of superfluous knowledge. What is often overlooked is the fact that we professors care about students off-the-shelf and enjoy it so much that we would never write the