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*Index* (continued)

$d$  the total number of variables in a system

$i$  usually, the index of an observation

$i$  the number of observations for variable  $i$

$n$  A vector of true values of length  $n$

$x$  Used to designate a single variable among a set of  $d$  variables

$x(t)$  Used to represent the state vector of the system at time  $t$ . In the case of the SIR models above we have  $x(t) = (S(t), I(t), R(t))$  and we will continue to refer to the constituent parts of the state vector by other names or by using subscripts as in  $x(t) = (x_1(t), x_2(t), x_3(t))$ . When the state vector  $x$  is viewed as a function of time, it will be described as the state trajectory.

$w_i$  A positive weight to be applied to the fitting terms of the  $i$ th variable

$D$  The derivative operator that transforms a function  $x$  into its time derivative  $dx/dt$ .  $D^n$  generates the derivative of order  $n$ .  $D^{-1}$  generates the antiderivative.  $Dx(t)$  is the vector of time derivatives of  $x(t)$ .

$a$  A coefficient or rate function in the homogeneous portion of a linear differential equation. This may be a function of time or the values of external variables, but may not be a function of the values of variables in the system.

$A$  A linear differential operator that transforms a function  $x$  into a linear combination of  $x$  and its derivatives  $D^k x$ . The coefficients in the linear combination can be functions of time, but may not be functions of values of variables.

$\theta$  A vector of parameters that require estimation from data