Contents

Preface v

1. Introduction 1

2. Vector Spaces, Bases, Linear Maps 5
   - 2.1 Motivations: Linear Combinations, Linear Independence and Rank 5
   - 2.2 Vector Spaces 18
   - 2.3 Indexed Families; the Sum Notation $\sum_{i \in I} a_i$ 26
   - 2.4 Linear Independence, Subspaces 32
   - 2.5 Bases of a Vector Space 39
   - 2.6 Matrices 47
   - 2.7 Linear Maps 53
   - 2.8 Linear Forms and the Dual Space 61
   - 2.9 Summary 64
   - 2.10 Problems 66

3. Matrices and Linear Maps 75
   - 3.1 Representation of Linear Maps by Matrices 75
   - 3.2 Composition of Linear Maps and Matrix Multiplication . 80
   - 3.3 Change of Basis Matrix 86
   - 3.4 The Effect of a Change of Bases on Matrices 90
   - 3.5 Summary 94
   - 3.6 Problems 94

4. Haar Bases, Haar Wavelets, Hadamard Matrices 101
4.1 Introduction to Signal Compression Using Haar Wavelets 101
4.2 Haar Bases and Haar Matrices, Scaling Properties of Haar Wavelets . . . 103
4.3 Kronecker Product Construction of Haar Matrices 108
4.4 Multiresolution Signal Analysis with Haar Bases 111
4.5 Haar Transform for Digital Images 113
4.6 Hadamard Matrices 122
4.7 Summary 125
4.8 Problems 125

5. Direct Sums, Rank-Nullity Theorem, Affine Maps 129
5.1 Direct Products 129
5.2 Sums and Direct Sums 130
5.3 The Rank-Nullity Theorem; Grassmann's Relation 136
5.4 Affine Maps 143
5.5 Summary 150
5.6 Problems 150

6. Determinants 159
6.1 Permutations, Signature of a Permutation 159
6.2 Alternating Multilinear Maps 164
6.3 Definition of a Determinant 168
6.4 Inverse Matrices and Determinants 177
6.5 Systems of Linear Equations and Determinants 180
6.6 Determinant of a Linear Map 183
6.7 The Cayley–Hamilton Theorem 183
6.8 Permanents 189
6.9 Summary 192
6.10 Further Readings 193
6.11 Problems 193

7. Gaussian Elimination, LU-Factorization, Cholesky Factorization, Reduced Row Echelon Form 199
7.1 Motivating Example: Curve Interpolation 199
7.2 Gaussian Elimination 203
7.3 Elementary Matrices and Row Operations 208
7.4 LU-Factorization 212

7.5 $PA = LU$ Factorization 218
7.6 Proof of Theorem 7.2 ⊛ 227
7.7 Dealing with Roundoff Errors; Pivoting Strategies 233
7.8 Gaussian Elimination of Tridiagonal Matrices 234
7.9 SPD Matrices and the Cholesky Decomposition 237
7.10 Reduced Row Echelon Form (RREF) 247
7.11 RREF, Free Variables, and Homogenous Linear Systems . 253
7.12 Uniqueness of RREF Form 257
7.13 Solving Linear Systems Using RREF 259
7.14 Elementary Matrices and Columns Operations 266
7.15 Transvections and Dilatations ⊛ 267
7.16 Summary 274
7.17 Problems 275

8. Vector Norms and Matrix Norms 287
8.1 Normed Vector Spaces 287
8.2 Matrix Norms 299
8.3 Subordinate Norms 304
8.4 Inequalities Involving Subordinate Norms 312
8.5 Condition Numbers of Matrices 314
8.6 An Application of Norms: Solving Inconsistent Linear Systems 323
8.7 Limits of Sequences and Series 325
8.8 The Matrix Exponential 328
8.9 Summary 331
8.10 Problems 333

9. Iterative Methods for Solving Linear Systems 339
9.1 Convergence of Sequences of Vectors and Matrices 339
9.2 Convergence of Iterative Methods 342
9.3 Description of the Methods of Jacobi, Gauss–Seidel, and Relaxation 344
9.4 Convergence of the Methods of Gauss–Seidel and Relaxation 353
9.5 Convergence of the Methods of Jacobi, Gauss–Seidel, and Relaxation for Tridiagonal Matrices 356
9.6 Summary 362
9.7 Problems 363

10. The Dual Space and Duality 367
10.1 The Dual Space E^* and Linear Forms . . . 367
10.2 Pairing and Duality Between E and E^* . . . 375
10.3 The Duality Theorem and Some Consequences . . . 380
10.4 The Bidual and Canonical Pairings . . . 386
10.5 Hyperplanes and Linear Forms . . . 388
10.6 Transpose of a Linear Map and of a Matrix . . . 389
10.7 Properties of the Double Transpose . . . 394
10.8 The Four Fundamental Subspaces . . . 397
10.9 Summary . . . 399
10.10 Problems . . . 400

11. Euclidean Spaces 405
11.1 Inner Products, Euclidean Spaces . . . 405
11.2 Orthogonality and Duality in Euclidean Spaces . . . 415
11.3 Adjoint of a Linear Map . . . 422
11.4 Existence and Construction of Orthonormal Bases . . . 425
11.5 Linear Isometries (Orthogonal Transformations) . . . 433
11.6 The Orthogonal Group, Orthogonal Matrices . . . 436
11.7 The Rodrigues Formula . . . 438
11.8 QR-Decomposition for Invertible Matrices . . . 441
11.9 Some Applications of Euclidean Geometry . . . 447
11.10 Summary . . . 448
11.11 Problems . . . 449

12. QR-Decomposition for Arbitrary Matrices 463
12.1 Orthogonal Reflections . . . 463
12.2 QR-Decomposition Using Householder Matrices . . . 469
12.3 Summary . . . 479
12.4 Problems . . . 480

13. Hermitian Spaces 487
13.1 Sesquilinear and Hermitian Forms, Pre-Hilbert Spaces and Hermitian Spaces . . . 487
13.2 Orthogonality, Duality, Adjoint of a Linear Map . . . 497
13.3 Linear Isometries (Also Called Unitary Transformations) . . . 504
13.4 The Unitary Group, Unitary Matrices . . . 505
13.5 Hermitian Reflections and QR-Decomposition . . . 509

13.6 Orthogonal Projections and Involutions . . . 514
13.7 Dual Norms . . . 517
13.8 Summary . . . 524
13.9 Problems . . . 525

14. Eigenvectors and Eigenvalues 531
14.1 Eigenvectors and Eigenvalues of a Linear Map . . . 531
14.2 Reduction to Upper Triangular Form . . . 540
14.3 Location of Eigenvalues . . . 545
14.4 Conditioning of Eigenvalue Problems . . . 549
14.5 Eigenvalues of the Matrix Exponential . . . 551
14.6 Summary . . . 554
14.7 Problems . . . 554

15. Unit Quaternions and Rotations in $\mathbf{SO}(3)$ 565
15.1 The Group $\mathbf{SU}(2)$ of Unit Quaternions and the Skew Field $\mathbb{H}$ of Quaternions . . . 566
15.2 Representation of Rotations in $\mathbf{SO}(3)$ by Quaternions in $\mathbf{SU}(2)$ . . . 567
15.3 Matrix Representation of the Rotation r_q . . . 572
15.4 An Algorithm to Find a Quaternion Representing a Rotation . . . 574
15.5 The Exponential Map $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$ . . . 578
15.6 Quaternion Interpolation $\circledast$ . . . 580
15.7 Nonexistence of a "Nice" Section from $\mathbf{SO}(3)$ to $\mathbf{SU}(2)$ . . . 583
15.8 Summary . . . 585
15.9 Problems . . . 585

16. Spectral Theorems in Euclidean and Hermitian Spaces 589
16.1 Introduction . . . 589
16.2 Normal Linear Maps: Eigenvalues and Eigenvectors . . . 589
16.3 Spectral Theorem for Normal Linear Maps . . . 595
16.4 Self-Adjoint, Skew-Self-Adjoint, and Orthogonal Linear Maps . . . 601
16.5 Normal and Other Special Matrices . . . 607
16.6 Rayleigh–Ritz Theorems and Eigenvalue Interlacing . . . 611
16.7 The Courant–Fischer Theorem; Perturbation Results . . . 615
16.8 Summary . . . 620

16.9 Problems . . . 620

17. Computing Eigenvalues and Eigenvectors 625
17.1 The Basic QR Algorithm . . . 627
17.2 Hessenberg Matrices . . . 635
17.3 Making the QR Method More Efficient Using Shifts . . . 641
17.4 Krylov Subspaces; Arnoldi Iteration . . . 647
17.5 GMRES . . . 651
17.6 The Hermitian Case; Lanczos Iteration . . . 652
17.7 Power Methods . . . 653
17.8 Summary . . . 656
17.9 Problems . . . 657

18. Graphs and Graph Laplacians; Basic Facts 659
18.1 Directed Graphs, Undirected Graphs, Incidence Matrices, Adjacency Matrices, Weighted Graphs . . . 662
18.2 Laplacian Matrices of Graphs . . . 670
18.3 Normalized Laplacian Matrices of Graphs . . . 675
18.4 Graph Clustering Using Normalized Cuts . . . 679
18.5 Summary . . . 682
18.6 Problems . . . 683

19. Spectral Graph Drawing 687
19.1 Graph Drawing and Energy Minimization . . . 687
19.2 Examples of Graph Drawings . . . 691
19.3 Summary . . . 696

20. Singular Value Decomposition and Polar Form 699
20.1 Properties of $f^* \circ f$ . . . 699
20.2 Singular Value Decomposition for Square Matrices . . . 703
20.3 Polar Form for Square Matrices . . . 706
20.4 Singular Value Decomposition for Rectangular Matrices . . . 709
20.5 Ky Fan Norms and Schatten Norms . . . 714
20.6 Summary . . . 715
20.7 Problems . . . 715

21. Applications of SVD and Pseudo-Inverses 719
21.1 Least Squares Problems and the Pseudo-Inverse . . . 719

21.2 Properties of the Pseudo-Inverse . . . 726
21.3 Data Compression and SVD . . . 733
21.4 Principal Components Analysis (PCA) . . . 735
21.5 Best Affine Approximation . . . 745
21.6 Summary . . . 752
21.7 Problems . . . 752

22. Annihilating Polynomials and the Primary Decomposition 755
22.1 Basic Properties of Polynomials; Ideals, GCD's . . . 757
22.2 Annihilating Polynomials and the Minimal Polynomial . . . 762
22.3 Minimal Polynomials of Diagonalizable Linear Maps . . . 764
22.4 Commuting Families of Diagonalizable and Triangulable Maps . . . 768
22.5 The Primary Decomposition Theorem . . . 771
22.6 Jordan Decomposition . . . 777
22.7 Nilpotent Linear Maps and Jordan Form . . . 780
22.8 Summary . . . 788
22.9 Problems . . . 789

Bibliography 791

Index 795